Asymptotics for eigenvalues of a non-linear integral system

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Abstract

Let $I = [a, b] \subset \mathbb{R}$, let $1 < q, p < \infty$, let u and v be positive functions with $u \in L_{p'}(I)$, $v \in L_q(I)$ and let $T : L_p(I) \to L_q(I)$ be the Hardy-type operator given by

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \ x \in I.$$

We show that the asymptotic behavior of the eigenvalues λ of the non-linear integral system

$$g(x) = (Tf)(x) \qquad (f(x))_{(p)} = \lambda(T^*(g_{(p)}))(x)$$

(where, for example, $t_{(p)} = |t|^{p-1}\mathrm{sgn}(t)$) is given by

$$\lim_{n\to\infty} n\hat{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r}, \text{ for } 1 < q < p < \infty.$$

$$\lim_{n\to\infty} n\check{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r}, \text{ for } \ 1 < p < q < \infty,$$

Here $r = 1/p' + 1/p$, $c_{p,q}$ is an explicit constant depending only on p and q, $\hat{\lambda}_n(T) = \max(sp_n(T, p, q))$, $\check{\lambda}_n(T) = \min(sp_n(T, p, q))$ where $sp_n(T, p, q)$ stands for the set of all eigenvalues λ corresponding to eigenfunctions g with n zeros.

1 Introduction and preliminaries

Through this paper we shall assume $I = [a, b]$, where $-\infty < a < b < \infty$, and let $p, q \in (1, \infty)$, $(x)_{(p)} := |x|^{p-1}$ sgn (x), $x \in \mathbb{R}$ and $1/p' = 1 - 1/p$.

Let u and v be positive functions on I, with $u \in L_{p'}(I)$, $v \in L_q(I)$.

Define the Hardy-type operator $T : L_p(I) \to L_q(I)$ by

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \ x \in I.$$

Such maps have been intensively studied: see [4, Chapter 2].

Since $|I| = b - a < \infty$, $u \in L_{p'}(I)$ and $v \in L_q(I)$ then T is compact, see [5, chapter 2].

As more detailed information about the native of the compactness of a map is provided by its approximation, Kolmogorov and Bernstein numbers, much attention has been paid to the asymptotic behavior of these numbers for the map T. The analysis is decidedly easier when $p = q$, and an account of the situation in this case is given in [5]. For the case $p \neq q$ we refer to [6], [7] in which a key role is played by the non-linear integral system:

$$g(x) = (Tf)(x) \tag{1.1}$$

and

$$(f(x))_{(p)} = \lambda(T^*(g_{(q)}))(x), \tag{1.2}$$

where $g_{(q)}$ is the function with value $(g(x))_{(q)}$ at x and T^* is the map defined by $(T^*f)(x) = u(x)\int_x^b v(y)f(y)dy$.

The non-linear system (1.1) and (1.2) gives us the following non-linear equation:

$$(f(x))_{(p)} = \lambda T^*((Tf)_{(q)})(x). \tag{1.3}$$

This is equivalent to its dual equation:

$$(s(x))_{(q')} = \lambda^* T((T^*s)_{(p')})(x). \tag{1.4}$$

And we have this relation: For given f and λ satisfying (1.3) we have $s = (Tf)_{(q)}$ and $\lambda^* = \lambda_{(p')}$ satisfying (1.4), and for given s and λ^* satisfying (1.4)we have $f = (T^*s)_{(p')}$ and $\lambda = \lambda^*_{(q)}$ satisfying (1.3).

By a spectral triple will be meant a triple (g, f, λ) satisfying (1.1) and (1.2), where $\|f\|_p = 1$; (g, λ) will be called a spectral pair; the function g corresponding to λ is called a spectral function and the number λ occurring in a spectral pair will be called a spectral number.

For the system (1.1) and (1.2) we denote by $SP(T, p, q)$ the set of all spectral triples; $sp(T, p, q)$ will stand for the set of all spectral numbers λ from $SP(T, p, q)$.

It can be seen that this non-linear system is related to the isoperimetric problem of determining

$$\sup_{g \in T(B)} \|g\|_q\,, \tag{1.5}$$

where $B := \{f \in L_p(I) : \|f\|_p \leq 1\}$.

Moreover, this problem can be seen as a natural generalization of the $p, q-$Laplacian differential equation. For if u and v are identically equal to 1 on I, then (1.1) and (1.2) can be transformed into the $p, q-$Laplacian differential equation:

$$-\left((w')_{(p)}\right)' = \lambda(w)_{(q)}, \tag{1.6}$$

with the boundary condition

$$w(a) = 0. \tag{1.7}$$

If g, f and λ satisfy (1.1) and (1.2) then, the integrals being over I,

$$\begin{aligned}\int |g(x)|^q dx &= \int g(g)_{(q)} dx = \int Tf(x)(g)_{(q)} dx \\ &= \int f(x) T^*(g)_{(q)} dx = \lambda^{-1} \int f(x)(f)_{(p)} \\ &= \lambda^{-1} \int |f(x)|^p dx.\end{aligned}$$

From this it follows that $\lambda^{-1} = \|g\|_q^q / \|f\|_p^p$ and then for $(g_1, \lambda_1) \in SP(T, p, q)$ we have $\lambda_1^{-1/q} = \|g_1\|_q$.

Given any continuous function f on I we denote by $Z(f)$ the number of distinct zeros of f on $\overset{o}{I}$, and by $P(f)$ the number of sign changes of f on this interval. The set of all spectral triples (g, f, λ) with $Z(g) = n$ $(n \in \mathbb{N}_0)$ will be denoted by $SP_n(T, p, q)$, and $sp_n(T, p, q)$ will represent the set of all corresponding numbers λ. We set $\hat{\lambda}_n = \max sp_n(T, p, q)$ and $\check{\lambda}_n = \min sp_n(T, p, q)$.

Our main result is that the asymptotic behavior of the $\hat{\lambda}_n$ can be determined when $1 < q < p < \infty$: we show that

$$\lim_{n \to \infty} n \hat{\lambda}_n(T) = c_{p,q} \left(\int_I (uv)^r dt \right)^{1/r},$$

where $r = 1/p' + 1/q$ and c_{pq} is a constant whose dependence on p and q is given explicitly. A corresponding result holds for $\check{\lambda}_n$ when $1 < p < q < \infty$. Moreover, $sp_n(T, p, p)$ contains exactly one element, so that in this case $\hat{\lambda}_n = \check{\lambda}_n = \lambda_n$ say, and the asymptotic behavior of the λ_n is given by the formula above.

We now give some results to prepare for the major theorems in §2 and §3.

Lemma 1.1 *Let $f \neq 0$ be a function on $[a, b]$ such that $Tf(a) = Tf(b) = 0$. Then $P(f) \geq 1$.*

Proof. This follows from the positivity of T and Rolle's theorem. ∎

Lemma 1.2 *Let $(g_i, f_i, \lambda_i) \in SP(T, p, q)$, $i = 1, 2$, $1 < p, q < \infty$. Then for any $\varepsilon > 0$,*

$$P(Tf_1 - \varepsilon Tf_2) \leq P(Tf_1 - \varepsilon^{(p-1)/(q-1)} (\lambda_2/\lambda_1)^{1/(q-1)} Tf_2). \tag{1.8}$$

If the function $f_1 - \varepsilon f_2$ has a multiple zero and $P(Tf_1 - \varepsilon^{(p-1)/(q-1)} (\lambda_2/\lambda_1)^{q/(q-1)} Tf_2) < \infty$, then the inequality (1.8) is strict.

Proof. We will use Lemma 1.1 and the fact that $\mathrm{sgn}(a-b) = \mathrm{sgn}((a)_{(p)}-(b)_{(p)})$.

$$\begin{aligned}
P(Tf_1 - \varepsilon Tf_2) &\le Z(Tf_1 - \varepsilon Tf_2) \le P(f_1 - \varepsilon f_2) \\
&\le P((f_1)_{(p)} - \varepsilon^{p-1}(f_2)_{(p)}) \\
&(\text{ use (1.3) for } f_1 \text{ and } f_2), \\
&\le P(\lambda_1 T^*((g_1)_{(q)}) - \varepsilon^{p-1}\lambda_2 T^*((g_2)_{(q)})) \\
&\le Z(\lambda_1 T^*((g_1)_{(q)}) - \varepsilon^{p-1}\lambda_2 T^*((g_2)_{(q)})) \\
&\le P((g_1)_{(q)} - \varepsilon^{p-1}(\lambda_2/\lambda_1)(g_2)_{(q)}) \\
&\le P(g_1 - \varepsilon^{(p-1)/(g-1)}(\lambda_2/\lambda_1)^{1/(q-1)} g_2) \\
&\le P(Tf_1 - \varepsilon^{(p-1)/(q-1)}(\lambda_2/\lambda_1)^{1/(q-1)} Tf_2).
\end{aligned}$$

■

Theorem 1.3 *For all $n \in \mathbb{N}$, $SP_n(T,p,q) \neq \emptyset$.*

Proof. This essentially follows ideas from [3] (see also [8]), but we give the details for the convenience of the reader. For simplicity we suppose that I is the interval $[0,1]$. A key idea in the proof is the introduction of an iterative procedure used in [3].

Let $n \in \mathbb{N}$ and define

$$\mathcal{O}_n = \left\{ z = (z_1, ..., z_{n+1}) \in \mathbb{R}^{n+1} : \sum\nolimits_{i=1}^{n+1} |z_i| = 1 \right\}$$

and

$$f_0(x,z) = sgn(z_j) \text{ for } \sum\nolimits_{i=0}^{j-1} |z_i| < x < \sum\nolimits_{i=1}^{j} |z_i|, \ j = 1, ..., n+1, \text{ with } z_0 = 0.$$

With $g_0(x,z) = Tf_0(x,z)$ we construct the iterative process

$$g_k(x,z) = Tf_k(x,z), \ f_{k+1}(x,z) = (\lambda_k(z) T^*(g_k(x,z))_{(q)})_{(p')},$$

where λ_k is a constant so chosen that

$$\|f_{k+1}\|_p = 1$$

and $1/p + 1/p' = 1$. Then, all integrals being over I,

$$\begin{aligned}
1 &= \int |f_k(x,z)|^p \, dx = \int f_k (f_k)_{(p)} dx = \int f_k \left([\lambda_{k-1} T^*((g_{k-1})_{(q)})]_{(p')} \right)_{(p)} dx \\
&= \int f_k \lambda_{k-1} T^*((g_{k-1})_{(q)}) dx \\
&= \lambda_{k-1} \int T(f_k)(g_{k-1})_{(q)} dx \le \lambda_{k-1} \|g_k\|_q \|g_{k-1}\|_q^{q-1}
\end{aligned}$$

and also

$$\begin{aligned}
\|g_{k-1}\|_q^q &= \int |g_{k-1}(x,z)|^q \, dx = \int (g_{k-1})_{(q)} g_{k-1} dx \\
&= \int (g_{k-1})_{(q)} T(f_{k-1}) dx = \int T^*((g_{k-1})_{(q)}) f_{k-1} dx \\
&= \lambda_{k-1}^{-1} \int \lambda_{k-1} T^*((g_{k-1})_{(q)}) f_{k-1} dx \\
&\le \lambda_{k-1}^{-1} \left(\int \left| (\lambda_{k-1} T^*((g_{k-1})_{(q)})_{(p')} \right|^{p'} dx \right)^{1/p'} \left(\int |f_{k-1}|^p \, dx \right)^{1/p} \\
&= \lambda_{k-1}^{-1} \left(\int \left| (\lambda_{k-1} T^*((g_{k-1})_{(q)})_{(p')} \right|^{p'} dx \right)^{1/p'} \\
&= \lambda_{k-1}^{-1} \left(\int |f_k|^p \, dx \right)^{1/p} = \lambda_{k-1}^{-1}.
\end{aligned}$$

From these inequalities it follows that

$$\|g_{k-1}(\cdot,z)\|_q \le \lambda_{k-1}^{-1/q} \le \|g_k(\cdot,z)\|_q \, .$$

This shows that the sequences $\{g_k(\cdot,z)\}$ and $\{\lambda_k^{-1/q}(z)\}$ are monotonic increasing. Put $\lambda(z) = \lim_{k\to\infty} \lambda_k(z)$; then $\|g_k(\cdot,z)\|_q \to \lambda^{-1/q}(z)$.

As the sequence $\{f_k(\cdot,z)\}$ is bounded in $L_p(I)$, there is a subsequence $\{f_{k_i}(\cdot,z)\}$ that is weakly convergent, to $f(\cdot,z)$, say. Since T is compact, $g_{k_i}(\cdot,z) \to Tf(\cdot,z) := g(\cdot,z)$ and we also have $f(\cdot,z) = (\lambda(z)T^*(g(\cdot,z))_{(q)})_{(p')}$. It follows that for each $z \in \mathcal{O}_n$, the sequence $\{g_{k_i}(\cdot,z)\}$ converges to a spectral function.

Now set $z = (0,0,...,0,1) \in \mathcal{O}_n$. Then $f_0(\cdot,z) = 1$, and as the operators T and T^* are positive, $g_k(\cdot,z) \ge 0$ for all k, so that $g(\cdot,z) \ge 0$. Thus $(g(\cdot,z), f(\cdot,z), \lambda(z)) \in SP_0(T,p,q) : SP_0(T,p,q) \ne \emptyset$.

Next we show that for all $n \in \mathbb{N}, SP_n(T,p,q) \ne \emptyset$. Given $n,k \in \mathbb{N}$, set

$$E_k^n = \{z \in \mathcal{O}_n : Z(g_k(\cdot,z)) \le n-1\}.$$

From the definition of T it follows that $g_k(\cdot,z)$ depends continuously on z; thus E_k^n is an open subset of $\mathcal{O}_n$ and $F_k^n := \mathcal{O}_n \backslash E_k^n$ is a closed subset of $\mathcal{O}_n$. Let $0 < t_1 < ... < t_n < 1$ and put

$$F_k(\alpha) = (g_k(t_1,\alpha), ..., g_k(t_n,\alpha)), \ \alpha \in \mathcal{O}_n.$$

Then F_k is a continuous, odd mapping from $\mathcal{O}_n$ to $\mathbb{R}^n$. By Borsuk's theorem, there is a point $\alpha_k \in \mathcal{O}_n$ such that $F_k(\alpha_k) = 0$; that is, $\alpha_k \in F_k^n$. From the definition of g_k and f_{k+1}, together with the positivity of T and T^*, we have

$$Z(g_{k+1}) \le P(f_{k+1}) \le Z(f_{k+1}) \le P(g_k) \le Z(g_k),$$

so that $E_k^n \subset E_{k+1}^n$, which implies that $F_k^n \supset F_{k+1}^n$. Hence there exists $\widetilde{\alpha} \in \cap_{k\ge1} F_k^n$, and as above we see that $g_k(\cdot,\widetilde{\alpha})$ converges, as $k \to \infty$, to a spectral

function $g(\cdot, \widetilde{\alpha}) \in SP_n(T,p,q)$. Thus $SP_n(T,p,q) \neq \emptyset$ and the proof is complete. ■

We note that the previous theorem is true for much more general integral operators (i.e. integral operators with totally positive kernel, see [8]).

We now define Kolmogorov widths $d_n(T)$ for T as a map from $L_p(I)$ to $L_q(I)$ when $1 < q, p < \infty$. These numbers are defined by:

$$d_n(T) = d_n = \inf_{X_n} \sup_{\|f\|_{p,I} \leq 1} \inf_{g \in X_n} \|Tf - g\|_{q,I} \,/\, \|f\|_{p,I}\,, \qquad n \in \mathbb{N}$$

where the infimum is taken over all n-dimensional subspaces X_n of $L_q(I)$.

To get an upper estimate for eigenvalues via the Kolmogorov numbers, we start by recalling the Makovoz lemma (see 3.11 in [3]).

Lemma 1.4 *Let $U_n \subset \{Tf; \|f\|_{p,I} \leq 1\}$ be a continuous and odd image of the sphere S^n in $\mathbb{R}^n$ endowed with the l_1 norm. Then*

$$d_n(T) \geq \inf\{\|x\|_{q,I}, x \in U_n\}$$

Lemma 1.5 *If $n > 1$, then $d_n(T) \geq \hat{\lambda}^{-1/q}$ where $\hat{\lambda} = \max\{\lambda \in \cup_{i=0}^n sp_i(p,q)\}$.*

Proof. Let us denote $\hat{\lambda} = \max\{\lambda \in \cup_{i=0}^n sp_i(p,q)\}$. The iteration process from the proof of Theorem 1.3 gives us for each $k \in \mathbb{N}$ and $z \in \mathcal{O}_n$ a function $g_k(.,z)$. By the Makavoz lemma we have

$$d_n(T) \geq \max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(.,z)\|_{q,I}. \tag{1.9}$$

Let us suppose that we have

$$\min_{z \in \mathcal{O}_n} \lim_{k \to \infty} \|g_k(.,z)\|_{q}, = \max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(.,z)\|_q. \tag{1.10}$$

Then from (1.9) and (1.10) it follows that

$$d_n(T) \geq \min_{z \in \mathcal{O}_n} \lim_{k \to \infty} \|g_k(.,z)\|_q \geq \hat{\lambda}^{-1/q},$$

since $\lim_{k\to\infty} g_k(.,z) \in SP(T,p,q)$. We have to prove (1.10). From the monotonicity of $\|g_k(.,z)\|_{q,I}$ we have

$$\max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(.,z)\|_q = \lim_{k \to \infty} \min_{z \in \mathcal{O}_n} \|g_k(.,z)\|_{q,}.$$

From max min $\leq$ min max it follows that

$$l := \lim_{k \to \infty} \min_{z \in \mathcal{O}_n} \|g_k(.,z)\|_{q,} = \max_{k \in \mathbb{N}} \min_{z \in \mathcal{O}_n} \|g_k(.,z)\|_q$$

$$\leq \min_{z \in \mathcal{O}_n} \max_{k \in \mathbb{N}} \|g_k(.,z)\|_q = \min_{z \in \mathcal{O}_n} \lim_{k \to \infty} \|g_k(.,z)\|_{q,} =: h$$

Denote $H_k(\varepsilon) = \{z \in \mathcal{O}_n; \|g_k(.,z)\|_q \leq h - \varepsilon\}$ where $0 < \varepsilon \leq h$.

Since the mapping $z \mapsto g_k(., z)$ is continuous, $H_k(\varepsilon)$ is a closed subset of $\mathcal{O}_n$, and from the construction of the sequence g_k we see that $H_0(\varepsilon) \supset H_1(\varepsilon) \supset ....$

If $y_0 \in \cap_{k\in\mathbb{N}} H_k(\varepsilon) \neq \emptyset$ then $h = \min_{z\in\mathcal{O}_n} \lim_{k\to\infty} \|g_k(., z)\|_q \leq \lim_{k\to\infty} \|g_k(., y_0)\|_q \leq h-\varepsilon$ is a contradiction. Then there exist $k_0 \in \mathbb{N}$ such that $H_k(\varepsilon) = \emptyset$ for $k \geq k_0$ and $\min_{z\in\mathcal{O}_n} \|g_k(., z)\|_q \geq h-\varepsilon$ for $k \geq k_0$. Then we have that $h = l$ and (1.10) is proved. ■

Next we define Bernstein widths which will help us in section 3. The Bernstein widths $b_n(T)$ for T: $L_p(I) \to L_q(I)$ when $1 < p, q < \infty$ are defined by:

$$b_n(T) := \sup_{X_{n+1}} \inf_{Tf\in X_{n+1}\setminus\{0\}} \|Tf\|_{q,I}/\|f\|_{p,I},$$

where the supremum is taken over all subspaces X_{n+1} of $T(L_p(I))$ with dimension $n+1$. Since u and v are positive functions, the Bernstein widths can be expressed as

$$b_n(T) = \sup_{X_{n+1}} \inf_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right)\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}},$$

where the supremum is taken over all $(n+1)$-dimensional subspaces $X_{n+1} = \mathrm{span}\{f_1, ..., f_{n+1}\} \subset L_p(I)$.

Now we use techniques from Theorem 1.3 to obtain an upper estimate for the Bernstein widths.

Lemma 1.6 *If $n > 1$ then $b_n(T) \leq \check{\lambda}^{-1/q}$, where $\check{\lambda} = \min(sp_n(p, q))$.*

Proof. Suppose there exists a linearly independent system of functions $\{f_1, ..., f_{n+1}\}$ on I, such that:

$$\min_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right)\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}} > \check{\lambda}^{-1/q}.$$

Let us define the $n-$dimensional sphere

$$O_n = \left\{T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right), \|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I} = 1\right\}.$$

Let $g_0(.) \in O_n$ and define a sequence of functions $h_k(.), g_k(.) = g_k(., g_0), k \in \mathbb{N}$, according to the following rule:

$$g_k(x) = Th_k(x), \qquad h_{k+1}(x) = (\lambda_k T^*(g_k(x))_{(q)})_{(p')},$$

where $\lambda_k > 0$ is a constant chosen so that $\|h_{k+1}\|_{p,I} = 1$.

We denote $O_n(k) = \{h_k(., h_0), h_0(.) \in O_n\}$. As in the proof of Theorem 1.3 we have:

$\|g_k\|_{q,I}$ is a nondecreasing as $k \nearrow \infty$. For each $k \in \mathbb{N}$ there exists $g_k \in O_n(k)$ with n zeros inside I; $\lim_{k\to\infty} g_k(., g_0)$ is an eigenfunction and there exists $g_0(.)$

such that $\lim_{k\to\infty} g_k(.,g_0)$ is an eigenfunction with n zeros. Moreover λ_k is monotonically decreasing as $k \nearrow \infty$.

Let $\overline{\alpha} \in \mathbb{R}^{n+1}$ be such that: $\overline{g_0}(.) = \left(\sum_{i=1}^{n+1} \overline{\alpha_i} f_i\right)$ is a function for which $\lim_{k\to\infty} \overline{g_k}(.,g_0)$ is an eigenfunction with n zeros.

Then we have the following contradiction:

$$\min_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|T\left(\sum_{i=1}^{n+1}\overline{\alpha_i} f_i\right)\|_{q,I}}{\|\sum_{i=1}^{n+1}\overline{\alpha_i} f_i\|_{p,I}} \le \|\overline{g_0}(.)\|_{q,I}$$
$$\le \lim_{k\to\infty} \|g_k(.,\overline{g_0}(.))\|_{q,I} \le \check{\lambda}^{-1/q},$$

■

In the next two sections we obtain an upper estimate for Kolmogorov numbers and a lower estimate for Bernstein numbers. We shall need the approximation numbers $a_n(T)$ of T, defined by $a_n(T) = \inf \|T - F\|$, where the infimum is taken over all linear operators F with rank at most $n-1$.

2 The case $q \le p$

We recall Jensen's inequality (see, for example [9], p.133) which will be of help in the next lemma.

Theorem 2.1 *If F is a convex function, and $h(.) \ge 0$ is a function such that $\int_I h(t)dt = 1$, then for every non-negative function g,*

$$F(\int_I h(t)g(t)dt) \le \int_I h(t)F(g(t))dt.$$

The following lemma give us a lower estimate for eigenvalues.

Lemma 2.2 *If $n > 1$ then $a_n(T) \le \widehat{\lambda}^{-1/q}$, where $\widehat{\lambda} = \max(sp_n(p,q))$.*

Proof. For the sake of simplicity we suppose that $|I| = 1$.

Let $(\widehat{g}, \widehat{f}, \widehat{\lambda}) \in SP_n(T,p,q)$. Denote by $\{a_i\}_{i=0}^n$ the set of zeros of $\widehat{g}$ (with $a_0 = a$) and by $\{b_i\}_{i=1}^{n+1}$ (with $b_{n+1} = b$) the set of zeros of $\widehat{f}$. Set $I_i = (b_i, b_{i+1})$ for $i = 1, ..., n$ and $I_0 = (a_0, b_1)$, and define

$$T_n f(x) := \sum_{i=0}^{n} \chi_{I_i}(.) v(.) \int_a^{a_i} u(t)f(t)dt.$$

Then the rank of T_n is at most n.

We have (see [4, Chapter 2]) $d_n(T) \le a_n(T) \le \sup_{\|f\|_p\le 1} \|Tf - T_n f\|_q$.

Let us consider the extremal problem:

$$\sup_{\|f\|_p\le 1} \|Tf - T_n f\|_q. \tag{2.1}$$

We can see that this problem is equivalent to

$$\sup\{\|Tf\|_q : \|f\|_p \leq 1, (Tf)(a_i) = 0 \text{ for } i = 0 \dots n\} \tag{2.2}$$

Since T and T_n are compact then there is a solution of this problem, that is, the supremum is attained. Let $\bar{f}$ be one such solution and denote $\bar{g} = T\bar{f}$. We can choose $\bar{f}$ such that $\bar{g}(t)\widehat{g}(t) \geq 0$, for all $t \in I$. We have $\|\bar{g}\|_{q,I} \geq \|\widehat{g}\|_{q,I}$

Note that for any $f \in L^p(I)$ such that $Tf(a_i) = 0$ for every $i = 0, ..., n$ we have $Tf(x) = T^+f(x)$ for each $x \in I$, where

$$T^+f(x) := \int_I K(x,t)f(t)dt = \sum_{i=0}^{n} \chi_{I_i}(.)v(.)\int_a^x u(t)f(t)dt$$

and

$$K(x,t) := \sum_{i=0}^{n} \chi_{I_i}(x)v(x)u(t)\chi_{(a_i,x)}\text{sgn}(x - a_i).$$

Set $s(t) = |\hat{g}(t)|^q\hat{\lambda}^q$, where $\hat{\lambda} = \|\hat{g}\|_{q,I}$. Then, all integrals being over I, we have

$$\begin{aligned}
\left(\int |\bar{g}(t)|^q dt\right)^{1/q} =& \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{\bar{g}(t)}{\hat{g}(t)}\right|^q dt\right)^{1/q} \\
& \text{(use Jensen's inequality, noting that } \int s(t)dt = 1) \\
\leq& \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{\bar{g}(t)}{\hat{g}(t)}\right|^p dt\right)^{1/p} \\
=& \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{T^+\bar{f}(t)}{\hat{g}(t)}\right|^p dt\right)^{1/p} \\
=& \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{\int K(t,\tau)\bar{f}(\tau)d\tau}{\hat{g}(t)}\right|^p dt\right)^{1/p} \\
=& \hat{\lambda}^{-1/q}\left(\int s(t)\left|\int \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)}\frac{\bar{f}(\tau)}{\hat{f}(\tau)}d\tau\right|^p dt\right)^{1/p} \\
& \text{(use Jensen's inequality, noting that} \\
& \left.\frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)} \geq 0 \text{ and } \int \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)}d\tau = 1\right) \\
\leq& \hat{\lambda}^{-1/q}\left(\int s(t)\int \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)}\left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p d\tau dt\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&=\hat{\lambda}^{-1/q}\left(\int\left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p \bar{f}(\tau)\int\frac{K(t,\tau)s(t)}{\hat{g}(t)}dtd\tau\right)^{1/p}\\
&=\hat{\lambda}^{-1/q}\left(\int\left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p \bar{f}(\tau)\int\frac{K(t,\tau)|\hat{g}(t)|^q}{\hat{g}(t)}\hat{\lambda}dtd\tau\right)^{1/p}\\
&=\hat{\lambda}^{-1/q}\left(\int\left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p \bar{f}(\tau)\int K(t,\tau)\hat{g}_{(q)}(t)dt\hat{\lambda}d\tau\right)^{1/p}\\
&\qquad\left(\text{ use }\int K(t,\tau)\hat{g}_{(q)}(t)dt\hat{\lambda}^q=\hat{\lambda}T^*(\hat{g}_{(q)})(t)=\hat{f}_{(p)}(t)\right)\\
&=\hat{\lambda}^{-1/q}\left(\int\left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p \hat{f}(\tau)\hat{f}_{(p)}(\tau)d\tau\right)^{1/p}\\
&\qquad(\text{ use }\hat{f}(t)\hat{f}_{(p)}(t)=|\hat{f}(t)|^p)\\
&=\hat{\lambda}^{-1/q}\left(\int\left|\bar{f}(\tau)\right|^p d\tau\right)^{1/p}=\hat{\lambda}^{-1/q}.
\end{aligned}
$$

From this it follows that $a_n(T)\le\hat{\lambda}^{-1/q}$. ■

Theorem 2.3 *If $1<q\le p<\infty$, then*

$$\lim_{n\to\infty} n\hat{\lambda}_n^{-1/q}=c_{pq}\left(\int_I |uv|^{1/r}dt\right)^r$$

where $r=1/p'+1/q$, $\hat{\lambda}_n=\max(sp_n(p,q))$ and

$$c_{pq}=\frac{(p')^{1/q}q^{1/p'}(p'+q)^{1/p-1/q}}{2B(1/q,1/p')}\tag{2.3}$$

(B denotes the Beta function).

Proof. From [7] we have

$$\lim_{n\to\infty} na_n(T)=\lim_{n\to\infty} nd_n(T)=c_{pq}\left(\int_I |uv|^{1/r}dt\right)^r$$

and since $d_n(T)\le a_n(T)$, $a_n(T)\searrow 0$ and $d_n(T)\searrow 0$ then from Lemma 2.2 follows:

$$c_{pq}\left(\int_I |uv|^{1/r}dt\right)^r\le\liminf_{n\to\infty} n\hat{\lambda}_n^{-1/q},$$

and from Lemma 1.5 we have

$$\limsup_{n\to\infty} n\hat{\lambda}_n^{-1/q}\le c_{pq}\left(\int_I |uv|^{1/r}dt\right)^r$$

which finishes the proof. ■

3 The case $p \le q$

Lemma 3.1 *Let $1 < p \le q < \infty$ and $n > 1$. Then $b_n(T) \ge \check{\lambda}^{-1/q}$, where $\check{\lambda} = \min(sp_n(p,q))$.*

Proof. We use the construction of Buslaev [2] Take $(\check{g}, \check{f}, \check{\lambda})$ from $SP_n(T,p,q)$ and denote by $a = x_0 < x_1 < ... < x_i < ... < x_n < x_{n+1} = b$ the zeros of $\check{g}$. Set $I_i = (x_{i-1}, x_i)$ for $1 \le i \le n+1$, $f_i(.) = \check{f}(.)\chi_{I_i}(.)$ and $g_i(.) = \check{g}(.)\chi_{I_i}(.)$. Then $Tf_i = g_i(.)$ for $1 \le i \le n+1$.

Define $X_{n+1} = \mathrm{span}\{f_1, ... f_{n+1}\}$. Since the supports of $\{f_i\}$ and $\{g_i\}$ are disjoint, then we have

$$b_n(T) \ge \inf_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\|T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right)\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}} = \inf_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\|\sum_{i=1}^{n+1} \alpha_i g_i\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}}.$$

We shall study the extremal problem of finding

$$\inf_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\|\sum_{i=1}^{n+1} \alpha_i g_i\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}}.$$

It is obvious that the extremal problem has a solution. Denote that solution by $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2, ...)$. Since $p \le q$, a short computation shows us that $\bar{\alpha}_i \ne 0$ for every i, moreover we can suppose that the $\bar{\alpha}_i$ alternate in sign. Label

$$\bar{\gamma} := \frac{\|\sum_{i=1}^{n+1} \bar{\alpha}_i g_i\|_{q,I}^q}{\|\sum_{i=1}^{n+1} \bar{\alpha}_i f_i\|_{p,I}^p};$$

then the solution of the extremal problem is given by $\bar{g} = \sum_{i=1}^{n+1} \bar{\alpha}_i g_i$, $\bar{f} = \sum_{i=1}^{n+1} \bar{\alpha}_i f_i$ where $\|\bar{f}\|_p = 1$.

Let us take the vector $\beta = (1, -1, ...)$. Define the functions $\tilde{g} = \sum_{i=1}^{n+1} \beta_i g_i$, $\tilde{f} = \sum_{i=1}^{n+1} \beta_i f_i$. Then

$$\lambda_n^{-1} := \frac{\|\sum_{i=1}^{n+1} \beta_i g_i\|_{q,I}^q}{\|\sum_{i=1}^{n+1} \beta_i f_i\|_{p,I}^p}.$$

It is obvious that $\bar{\gamma} \le \lambda_n^{-1}$. Suppose that $\bar{\gamma} < \lambda^{-1}$.

Since $\bar{\alpha}_i \ne 0$, $|\beta_i| = 1$ and $\bar{\gamma} < \lambda^{-1}$ then $0 < \varepsilon^* := \min_{1 \le i \le n+1}(\beta_i/\bar{\alpha}_i) < 1$. From Lemma 1.2 follows

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \le P(T(\tilde{f}) - \varepsilon^{*(p-1)/(q-1)}(\bar{\gamma}/\lambda_n^{-1})^{1/(q-1)} T(\bar{f})).$$

By repeated use of Lemma 1.2 with the help of $(\varepsilon^*)^{(p-1)/(q-1)} \le \varepsilon^* < 1$ and $\bar{\gamma}/\lambda^{-1} < 1$ we get

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \le P(T(\tilde{f})) = n.$$

On the other hand we have from Lemma 1.1 and the definition of ε^* that

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \le P(\tilde{f} - \varepsilon^* \bar{f}) = P(\sum_{i=1}^{n+1} \beta_i f_i - \varepsilon^* \sum_{i=1}^{n+1} \bar{\alpha}_i f_i) \le n - 1,$$

which contradicts $\bar{\gamma} < \lambda^{-1}$. ■

Theorem 3.2 *If $1 < p \leq q < \infty$ then*

$$\lim_{n\to\infty} n\check{\lambda}_n^{-1/q} = c_{pq}\left(\int_I |uv|^r dt\right)^{1/r}$$

where $r = 1/p' + 1/q$, $\check{\lambda}_n = \min(sp_n(p,q))$ and c_{pq} as in (2.3).

Proof. From [6] we have

$$\lim_{n\to\infty} nb_n(T) = c_{pq}\left(\int_I |uv|^r dt\right)^{1/r}$$

and since $b_n(T) \searrow 0$ then from Lemma 1.6 it follows that

$$c_{pq}\left(\int_I |uv|^r dt\right)^{1/r} \leq \liminf_{n\to\infty} n\check{\lambda}_n^{-1/q}.$$

Moreover, from Lemma 3.1 we have

$$\limsup_{n\to\infty} n\check{\lambda}_n^{-1/q} \leq c_{pq}\left(\int_I |uv|^r dt\right)^{1/r}$$

which finishes the proof. ■

When $p = q$ the following lemma follows from Theorem 2.3 and Theorem 3.2 (we can find this result in a sharper form in [1]).

Remark 3.3 *When $p = q$ then*

$$\lim_{n\to\infty} n{\lambda_n}^{-1/q} = c_{pq}\left(\int_I |uv|^r dt\right)^{1/r}$$

where $r = 1/p' + 1/q$, c_{pq} as in (2.3) and λ_n is the single point in $sp_n(p,q)$.

References

[1] C. Bennewitz, Approximation numbers = Singular values, Journal of Computational and Applied Mathematics, to appear.

[2] A.P. Buslaev, On Bernstein-Nikol'skiĭ inequalities and widths of Sobolev classes of functions, Dokl. Akad. Nauk **323** (1992), no. 2, 202-205.

[3] A.P. Buslaev and V.M. Tikhomirov, Spectra of nonlinear differential equations and widths of Sobolev classes. Mat. Sb. **181** (1990), 1587-1606; English transl. in Math. USSR Sb. **71** (1992), 427-446.

[4] Edmunds, D.E. and Evans, W.D., *Spectral theory and differential operators,* Oxford University Press, Oxford, 1987.

[5] D.E. Edmunds and W.D. Evans, *Hardy operators, function spaces and embeddings,* Springer, Berlin-Heidelberg-New York, 2004.

[6] D.E. Edmunds and J. Lang, Bernstein widths of Hardy-type operators in a non-homogeneous case, to be published in Journal of Mathematical Analysis and Applications.

[7] D.E. Edmunds and J. Lang, Approximation numbers and Kolmogorov widths of Hardy-type operators in a non-homogeneous case, Mathematische Nachrichten, **279** (2006), no. 7, 727-742.

[8] Nguen, T'en Nam, The spectrum of nonlinear integral equations and widths of function classes, Math. Notes **53** (1993), no. 3-4, 424-429.

[9] A. Kufner, O. John and S. Fucik, *Function spaces, Noordhoff International Publishing, Leyden, (1977).*

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